Hybrid Finite Element-Boundary Integral Method Accelerated by the NSPW-MLFMA

Pieterjan Demarcke*, Hendrik Rogier

Department of Information Technology, Ghent University
Sint-Pietersnieuwstraat 41, 9000 B-Gent, Belgium
pieterjan.demarcke@intec.ugent.be

***Abstract*—This article presents a hybrid finite element-boundary integral equation (FE-BIE) method where the boundary integral interactions containing the 2D Green's kernel function are accelerated by the nondirective stable plane wave multilevel fast multipole algorithm (NSPW-MLFMA). This hybrid method enables the fast simulation of very large scale scattering problems with multiple homogeneous and inhomogeneous dielectrics and perfectly electric conducting (PEC) objects. The new hybrid technique with FMM acceleration applies for both high frequency as low frequency as long as the finite element mesh is sufficiently fine to contain the numerical dispersion within the desired accuracy. The hybrid formulation is outlined, and its validity is demonstrated by means of a 2D scattering problem.**

I. Introduction

In the field of computational electromagnetics, several numerical techniques can be used to obtain a full-wave solution to the time-harmonic set of Maxwell's equations. The BIE formulation, discretized into a system of linear equations by the method of moments (MoM), and the FE method are both very popular approaches, and depending on problem size and material properties, one of these is preferred over the other. The strength of the BIE method lies in the fact that for homogeneous media, only the boundaries must be discretized to find a full-wave solution for the complete domain. Accelerating the BIE method with an MLFMA, the complexity can be reduced to $O(N\log(N))$ or even $O(N)$. Here, a NSPW-MLFMA [1], [2] is used to accelerate the iterative solution. Compared to a regular MLFMA, this method remains stable at low frequencies, including the DC-limit. This way, structures with fine details compared to the wavelength can still be accurately modeled and simulated.

The BIE formulation cannot be applied for arbitrary inhomogeneous and/or anisotropic structures, since, in general, no analytical expression for the Green's function is available. In this case, FEM can be used to discretize the domain into small elements of constant material parameters (ϵ_r, μ_r). The discretization of the complete domain results in a large number of unknowns, but the resulting system matrix is sparse and can be compacted by means of a multifrontal algorithm to eliminate the interior unknowns. The complexity of one matrix-vector product is $O(N^2)$, with N the number of unknowns at the boundary, both for the sparse system and the compacted system.

Coupling the FE method with an accelerated, LF stable BIE method enables the simulation of large and multiscale problems with small geometrical details and inhomogeneous as well as anisotropic regions. The OpenFMM [3] project uses an efficient, asynchronous parallelization scheme to implement the NSPW-MLFMA, and is used here as a starting framework for the hybrid FE-BIE method. After outlining the general formulation, the coupling between the FE and BIE method is discussed together with some numerical properties of the discretized hybrid system. The new hybrid method is validated by means of an example involving a single dielectric scatterer, while more advanced results demonstrating the true power of the hybrid algorithm will be shown at the time of the conference.

II. General Formulation

We consider a linear, homogeneous and isotropic background medium $\mathcal{M}_0$ characterized by an electric permittivity ϵ_0 and a magnetic permeability μ_0. Embedded in this medium are a number of cylindrical dielectric and perfectly electric conducting (PEC) objects, aligned along the z-axis and each characterized by arbitrary parameters $\bar{\bar{\epsilon}}_i$ and $\bar{\bar{\mu}}_i$. In general, $\bar{\bar{\epsilon}}_i$ and $\bar{\bar{\mu}}_i$ are 3×3 tensors that exhibit frequency and position dependency. Here, in the 2D case, we restrict ourselves to materials which are reciprocal and invariant along the z-direction, with the anisotropy limited to the xy-plane.

For each object, its cross-section S_i in the xy-plane is bounded by the closed contour ∂S_i, on which we define the local standard coordinate system $(\mathbf{n}_i, \mathbf{t}_i, \mathbf{z})$ with $\mathbf{n}_i$ pointing outwards, as shown in Fig. 1. The objects are illuminated by an incoming electromagnetic field $(\mathbf{e}^{\text{in}}, \mathbf{h}^{\text{in}})$ present in the background medium $\mathcal{M}_0$. We assume time-harmonic fields with a $e^{j\omega t}$ time dependency. To simplify the expressions we will focus on pure transverse magnetic (TM) excitations (no z-dependency in the fields), but the formulations can be extended to coupled 2.5D TE/TM problems with arbitrary fields after applying the spatial Fourier transform along the z-direction to decompose them into spectral components with a z-dependency $e^{-j\beta z}$.

A. 2D Boundary Integral Formulation

In the BIE formulation, fields at an observation point $\mathbf{r}$ inside a homogeneous object are related to the fields at the boundary by means of an integral equation containing the Green's function $G_i(\mathbf{r}' - \mathbf{r})$ as integration kernel. This Green's

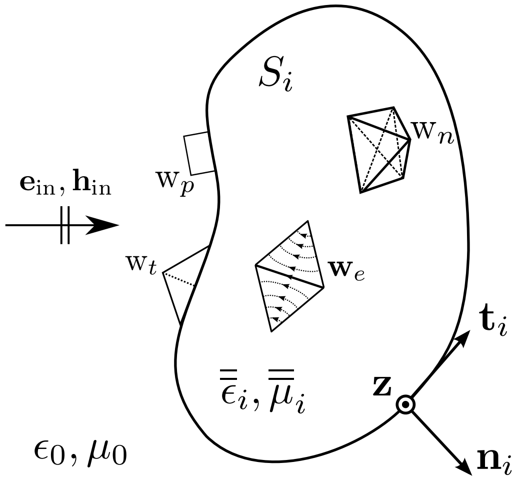


Fig. 1: Cyclindrical scatterer with cross-section S_i and local coordinate system $(\mathbf{n}_i, \mathbf{t}_i, \mathbf{z})$ defined on its boundary. Edge basis functions $\mathbf{w}_e$ and node basis functions w_n are defined on the triangulation of S_i, and tent basis functions w_t and pulse basis functions w_p are defined on the segments at the boundary ∂S_i.

function is the fundamental solution of the Helmholtz equation inside the homogeneous isotropic medium, and in 2D, we have

$$G_i(\mathbf{r}' - \mathbf{r}) = \frac{j}{4} H_0^{(2)}(k_i \, |\mathbf{r}' - \mathbf{r}|), \tag{1}$$

with $H_0^{(2)}$ the 0-th order Hankel function of the second kind, $k_i = \omega\sqrt{\epsilon_i \mu_i}$ and (ϵ_i, μ_i) the scalar permittivity and scalar permeability of the medium, respectively.

To solve a scattering problem involving multiple homogeneous dielectric and PEC objects, it is sufficient to know the fields at the boundaries. After moving the observation point $\mathbf{r}$ towards the boundary and taking the limit, continuity of the tangential $\mathbf{e}$ and $\mathbf{h}$ fields can be expressed at each boundary. Specifically, for the 2D TM case, the fields at the boundary ∂S_i satisfy the coupled set of integral equations

$$\begin{aligned} \mathrm{e}_{z,i}^{\mathrm{in}} &= \frac{1}{2}\mathrm{e}_z - \int\limits_{\partial S_i} \left[\mathrm{e}_z \frac{\partial G_i}{\partial \mathbf{n}_i} - j\omega\mu_i \mathrm{h}_{t,i} G_i \right] \mathrm{d}c' \\ \mathrm{h}_{t,i}^{\mathrm{in}} &= \frac{1}{2}\mathrm{h}_{t,i} - \int\limits_{\partial S_i} \left[\frac{-j\omega\epsilon_i}{k_i^2} \mathrm{e}_z \frac{\partial^2 G_i}{\partial \mathbf{n}_i \partial \mathbf{n}_i'} - \mathrm{h}_{t,i} \frac{\partial G_i}{\partial \mathbf{n}_i} \right] \mathrm{d}c', \end{aligned} \tag{2}$$

with $\mathbf{h}_t^{\mathrm{in}}$ and $\mathrm{e}_z^{\mathrm{in}}$ the source fields present inside S_i, and $\mathrm{h}_{t,i}$ the tangential magnetic field $\mathbf{h}_t \cdot \mathbf{t}_i$. Special care is needed for the singularity at $\mathbf{r} = \mathbf{r}'$, and therefore the integrals in (2) correspond to their principal value. Using the PMCHWT formulation, the coupled integral equations (2) of two adjacent homogeneous objects are combined into a system of first-kind integral equations.

B. 2D Finite Element Formulation

The finite element formulation consists of an electromagnetic boundary value problem (BVP) where the boundary conditions are the tangential electric and magnetic fields at the boundaries. The formulation is only valid in bounded domains, so in the background medium the BIE formulation of the previous section will be used to enforce the radiation condition. Starting from the time-harmonic curl Maxwell equations and replacing the electric and magnetic flux densities $\mathbf{d}$ and $\mathbf{b}$ with their appropriate constitutive relations, the weak formulation of the electric and magnetic BVP is obtained by weighting both equations with edge element basis functions and integrating over the domain. Specifically, in the 2D TM case we have

$$\int\limits_S \left[\nabla_{\mathrm{t}} \times \mathrm{w}_n \mathbf{z} \cdot \overline{\overline{\mu}}_r^{-1} \cdot \nabla_{\mathrm{t}} \times \mathrm{e}_z \mathbf{z} - k_0^2 \mathrm{w}_n \epsilon_{r,zz} \mathrm{e}_z \right] \mathrm{d}S = j\omega\mu_0 \int\limits_{\partial S} \mathbf{h}_t \cdot \mathbf{t} \mathrm{w}_n \, \mathrm{d}c \tag{3}$$

for the electric field formulation (EFF) and

$$\int\limits_S \left[\nabla_{\mathrm{t}} \times \mathbf{w}_e \cdot \overline{\overline{\epsilon}}_r^{-1} \cdot \nabla_{\mathrm{t}} \times \mathbf{h}_t - k_0^2 \mathbf{w}_e \cdot \overline{\overline{\mu}}_r^{-1} \cdot \mathbf{h}_t \right] \mathrm{d}S = j\omega\epsilon_0 \int\limits_{\partial S} \mathrm{e}_z \mathbf{w}_e \cdot \mathbf{t} \, \mathrm{d}c \tag{4}$$

for the magnetic field formulation (MFF). The node basis functions w_n and the edge basis functions $\mathbf{w}_e$ are shown in Fig. 1 and form a basis for the electrical field e_z and the transversal magnetic field $\mathbf{h}_t$, respectively.

III. Hybrid System

To numerically obtain the full-wave solution of a 2D scattering problem involving multiple objects, a triangular mesh is constructed for the inhomogeneous and anisotropic regions, while the boundaries of the homogeneous and PEC objects are properly segmented to achieve the required accuracy. For the BIE formulation, a segment length of $\frac{\lambda}{10}$ is usually sufficient, but frequently smaller segments are needed to retain the desired geometric detail, especially in multiscale problems. To achieve the same accuracy with the FE formulation, the average element size in the mesh typically needs to be smaller. This effect is due to discretizing the entire cross-section of the subdomain, giving rise to numerical dispersion, which originates from the difference between the numerical and the exact wave number inside the medium. As a result, the segment length at the boundary between a FE and BIE region will be even smaller. Traditional MLFMA methods suffer from an LF breakdown when the segment length becomes too small, but since the NSPW-MLFMA remains stable at LF, the interactions between those small segments can still be accelerated accurately.

A. BIE discretization

For the BIE formulation, the method of moments (MoM) is used to discretize (2). Tent basis functions w_t and pulse basis functions w_p are defined on the segments at the boundary (see Fig. 1) and are used to decompose e_z and h_t, respectively. After testing the BIEs with the appropriate test functions ($\mathrm{e}_z^{\mathrm{in}}$

is tested with w_p, $\mathrm{h}_t^{\mathrm{in}}$ with w_t), the following linear system is obtained:

$$\begin{bmatrix} \mathrm{E}_i^{\mathrm{in}} \\ -\mathrm{H}_i^{\mathrm{in}} \end{bmatrix} = \left(\begin{bmatrix} \mathrm{Z}_{\mathrm{EH}i} & \mathrm{Z}_{\mathrm{EE}i} \\ -\mathrm{Z}_{\mathrm{HH}i} & -\mathrm{Z}_{\mathrm{HE}i} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & \mathrm{D}_{\mathrm{E}i} \\ -\mathrm{D}_{\mathrm{H}i} & 0 \end{bmatrix} \right) \begin{bmatrix} \mathrm{H}_i \\ \mathrm{E}_i \end{bmatrix}. \tag{5}$$

B. FE discretization

The FE system matrices corresponding to the EFF (3) and MFF (4) follow after decomposing the e_z and $\mathbf{h}_t$ fields into a linear combination of node and edge basis functions, respectively. For the EFF, the sparse system matrix Z_n contains the interactions between the different overlapping nodal basis functions, and since the unknowns corresponding to the internal nodes do not interact with unknowns from other FE or BIE regions, they can be eliminated by compressing the matrix Z_n into Z_n^c by means of a multifrontal algorithm [4]. The same is performed for the MFF, which results in two decoupled linear systems relating the electrical field e_z and the magnetic field h_t at the boundary:

$$\begin{bmatrix} c_1\mathrm{Z}_n^c & 0 \\ 0 & c_2\mathrm{Z}_e^c \end{bmatrix} \begin{bmatrix} \mathrm{E}_i \\ \mathrm{H}_i \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{D}_{\mathrm{H}i} \\ \mathrm{D}_{\mathrm{E}i} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{E}_i \\ \mathrm{H}_i \end{bmatrix} \tag{6}$$

The coefficients c_1 and c_2 are used to compensate for scaling after projecting the FE basis functions on the boundary and matching them with their BIE counterpart.

C. Coupling

The correct ordering of the known and unknown fields in (5) makes the interaction matrix with entries Z_{XY} symmetric. From the Petrov-Galerkin scheme it follows that $\mathrm{D}_{\mathrm{E}i} = \mathrm{D}_{\mathrm{H}i}^{\mathrm{T}}$, so after the appropriate coupling with the EFF or MFF system matrix, a fully symmetric hybrid system is obtained. Here we will only use the MFF (4), and the final symmetric hybrid system is

$$\begin{bmatrix} \mathrm{E}_i^{\mathrm{in}} \\ -\mathrm{H}_i^{\mathrm{in}} \end{bmatrix} = \left(\begin{bmatrix} \mathrm{Z}_{\mathrm{EH}i} & \mathrm{Z}_{\mathrm{EE}i} \\ -\mathrm{Z}_{\mathrm{HH}i} & -\mathrm{Z}_{\mathrm{HE}i} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2c_2\mathrm{Z}_e^c & \mathrm{D}_{\mathrm{E}i} \\ \mathrm{D}_{\mathrm{H}i} & 0 \end{bmatrix} \right) \begin{bmatrix} \mathrm{H}_i \\ \mathrm{E}_i \end{bmatrix}. \tag{7}$$

D. Numerical Solution of the Hybrid System

The hybrid system (7) is solved using an iterative approach where one matrix-vector product is split into two parts. For the BIE part, the NSPW-MLFMA reduces the complexity of one matrix-vector product to $O(N\log(N))$ or $O(N)$. The algorithm is known to be error controllable and stable at LF, and various preconditioning techniques can be used to further speed up the convergence of the iterative solution process. For the FE part, the complexity remains $O(N^2)$ with N the number of unknowns at the boundary, both for the sparse and for the compressed representation. In terms of accuracy, the general trend is that the error decreases when the grid is refined, without problems such as the LF breakdown of the classic MLFMA. Special care must be taken to properly condition the FE system. For a more detailed analysis we refer to [5], but in general the convergence of the iterative process worsens when the grid is refined or when the frequency is increased. Again, different preconditioning techniques are available. For the example in the next section, no preconditioning is applied since the number of unknowns is too small to benefit from any improvement in conditioning.

IV. Example

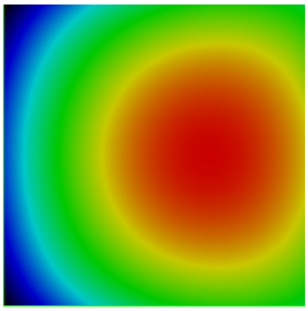
To validate the hybrid formulation and its implementation, the scattering of a plane wave at a single homogeneous dielectric rectangular object is studied. The scatterer has a relative dielectric permittivity $\epsilon_r = 2.0$ and is embedded in free space. The sides of the $2 \times 2\,\mathrm{m}$ rectangular cross section are aligned along the x and y axis. The boundary is discretized into 192 segments, and the interior mesh for the FE part contains 3824 triangles. The TM plane wave travels along the positive x direction once at $10\,\mathrm{MHz}$, once at $200\,\mathrm{MHz}$ and once at $400\,\mathrm{MHz}$. At each frequency, the problem is solved with a reference BIE method and with the proposed FE-BIE method, which are both accelerated by the NSPW-MLFMA.

At $10\,\mathrm{MHz}$, the relative length of a boundary segment is around $\frac{\lambda}{500}$. The maximum interaction distance between two segments is almost $\frac{\lambda}{10}$, so a classic MLFMA algorithm can only accelerate the most distant interactions before the LF breakdown occurs. The NSPW-MLFMA does not suffer from this limitation, and the algorithm successfully accelerates the problem using an FMM tree with 4 LF levels and 1 HF level. The single HF level corresponds to the only available level in the classic MLFMA approach. As for the field inside the scatterer at $10\,\mathrm{MHz}$, the e_z component of the BIE solution (Fig. 2a) is identical to the e_z component of the FE-BIE solution (Fig. 2b). The same correspondence is found between the h_t fields at the boundary, as can be seen in Fig. 3a. The FE part of the FE-BIE algorithm accurately solves the interior field distributions, even at element sizes of order $\frac{\lambda}{500}$. By applying the NSPW-MLFMA, the BIE interactions can still be accelerated, resulting in a fast and accurate solution.

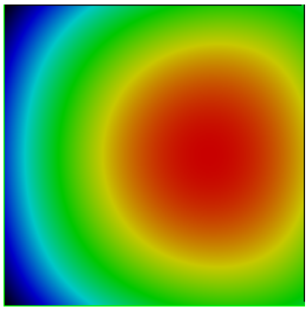
When the frequency is increased to $200\,\mathrm{MHz}$, the NSPW-MLFMA still uses 4 LF FMM levels and 1 HF FMM level to speed up the BIE interactions. The solutions of the reference method and the FE-BIE method are still identical, which is shown in Fig. 2c and Fig. 2d for the e_z field distribution and in Fig. 3b for the tangential h_t field on the boundary. At this frequency, the FE mesh seems sufficiently fine to accurately resolve the phase of the fields inside the object, and no dispersion error is observed.

Further increasing the frequency to $400\,\mathrm{MHz}$, no LF interactions remain, and the FMM tree now consists of 5 HF levels. The mesh element size approaches $\frac{\lambda}{10}$, and a small dispersion error starts to show. Comparing Fig. 2e and Fig. 2f, we see that the interior field distribution of the FE-BIE solution starts to deviate somewhat from the reference solution due to this dispersion error. The same can be observed for the h_t fields at the boundary in Fig. 3c between segment index 160 and 180. To obtain a more accurate solution, the mesh must be refined.

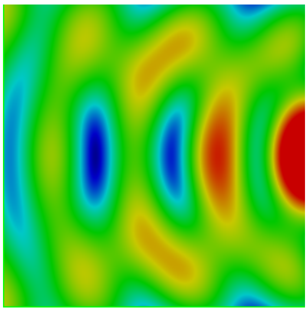
The example validates the hybrid formulation, and as long as the mesh element size is sufficiently small compared to the wavelength, fast and accurate results can be obtained without any restriction on frequency or segment size.



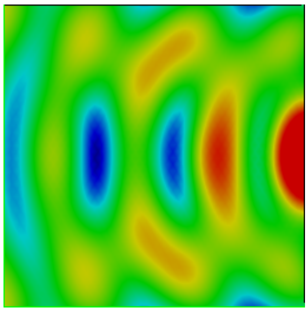
(a) f = 10 MHz, BIE method



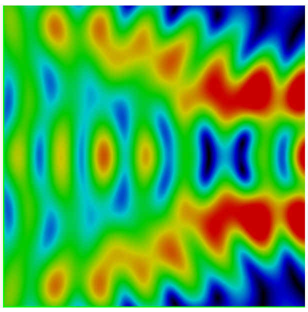
(b) f = 10 MHz, FE-BIE method



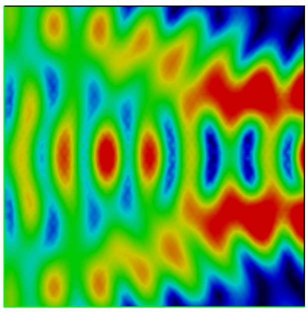
(c) f = 200 MHz, BIE method



(d) f = 200 MHz, FE-BIE method

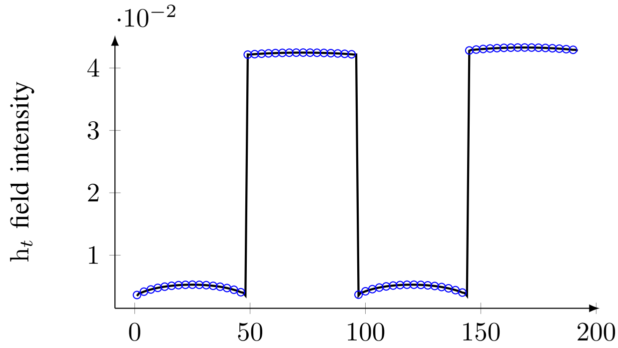


(e) f = 400 MHz, BIE method

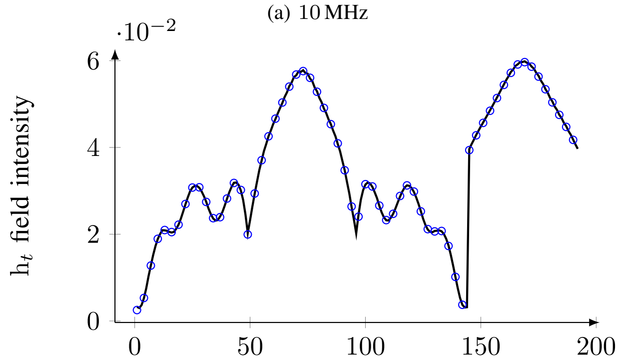


(f) f = 400 MHz, FE-BIE method

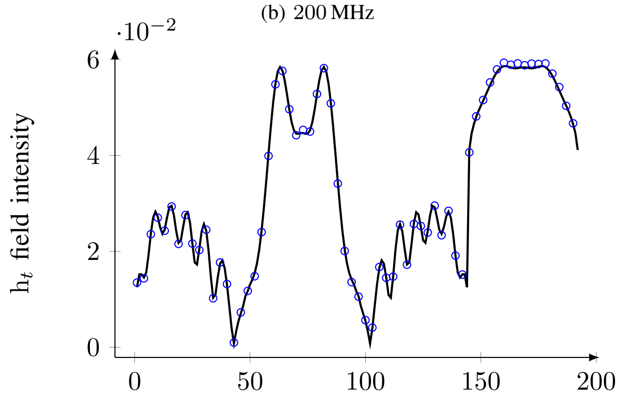
Fig. 2: Comparison between the BIE (left) and hybrid FE-BIE (right) solution of the vertical electric field inside a rectangular homogeneous dielectric medium ($\epsilon_r = 2.0, \mu_r = 1.0$) embedded in free space and excited with a plane wave impinging from the left at 10 MHz (top), at 200 MHz (middle) and at 400 MHz (bottom).



(a) 10 MHz



(b) 200 MHz



(c) 400 MHz

Fig. 3: Magnetic field intensity as a function of segment index for the BIE (line) and the hybrid FE-BIE (circles) solution at 10 MHz (top), at 200 MHz (middle) and at 400 MHz (bottom).

V. Conclusion

Combining the NSPW-MLFMA accelerated BIE method and the FE method yields a powerful hybrid algorithm that can solve multiscale scattering problems with anisotropic and inhomogeneous objects within a controlled accuracy. The LF stability of the NSPW-MLFMA enables the acceleration of the BIE interactions without restrictions on segment length or operating frequency. This way, complex structures with multiple homogeneous and inhomogeneous regions can be modelled to any desired geometric accuracy while retaining the $O(N \log(N))$ complexity in the BIE interactions. A reference example validated the algorithm both in the LF and the HF regime and demonstrated the advantages over the classic MLFMA approach. Real-life scattering problems can now be solved to demonstrate the power of the method, and they will be presented at the time of the conference.

References

[1] I. Bogaert, D. Pissoort, and F. Olyslager, "A normalized plane wave method for 2-D Helmholtz problems," *Microwave and Optical Technology Letters*, vol. 48, no. 2, pp. 237–243, February 2006.

[2] I. Bogaert, J. Peeters, and F. Olyslager, "A nondirective plane wave MLFMA stable at low frequencies," *IEEE Transactions on Antennas and Propagation*, vol. 56, no. 12, pp. 3752–3767, December 2008.

[3] J. Fostier and F. Olyslager, "An asynchronous parallel MLFMA for scattering at multiple dielectric objects," *Antennas and Propagation, IEEE Transactions on*, vol. 56, no. 8, pp. 2346–2355, August 2008.

[4] I. S. Duff and J. K. Reid, "The multifrontal solution of indefinite sparse symmetric linear equations," *ACM Trans. Math. Softw.*, vol. 9, no. 3, pp. 302–325, 1983.

[5] A. C. Zhu, Y. Cangellaris, *Multigrid Finite Element Methods for Electromagnetic Field Modeling*. John Wiley & Sons, 2006.